Archive for Mathematical Logic manuscript No.
(will be inserted by the editor)

Well quasi orders in a categorical setting

Marco Benini · **Roberta Bonacina**

Received: date / Accepted: date

Abstract This article describes well quasi orders as a category, focusing on limits and colimits. In particular, while quasi orders with monotone maps form a category which is finitely complete, finitely cocomplete, and with exponentiation, the full subcategory of well quasi orders is finitely complete and cocomplete, but with no exponentiation. It is interesting to notice how finite antichains and finite proper descending chains interact to induce this structure in the category: in fact, the full subcategory of quasi orders with finite antichains has finite colimits but no products, while the full subcategory of well founded quasi orders has finite limits but no coequalisers.

Moreover, the article characterises when exponential objects exist in the category of well quasi orders and well founded quasi orders. This completes the systematic description of the fundamental constructions in the categories of quasi orders, well founded quasi orders, quasi orders with finite antichains, and well quasi orders.

Keywords well quasi order · well founded quasi order · preorder categories · exponentiation

Mathematics Subject Classification (2000) 06A11 · 06A75 · 03G10 · 18C10

1 Introduction

Well quasi orders are an important class of order structures, with many applications, see [4, 8] as a starting point. They have been widely studied and rediscovered many times [6] before reaching a stable status. Apparently, they have never been systematically studied from a categorical point of view: although quasi orders have a simple and neat formalisation as categories, which led to many insights and applications, well quasi orders have been neglected in this respect. The present paper wants to

Dipartimento di Scienza e Alta Tecnologia
Università degli Studi dell'Insubria
E-mail: marco.benini@uninsubria.it · E-mail: r.bonacina1@uninsubria.it

overcome this missing point in the literature, systematically showing how the category of well quasi orders with monotone maps behaves, focusing on the existence and construction of limits, colimits and exponential objects—subobject classifiers are known not to exist. Although most of the presented results are well known, the contributions about coequalisers and exponential objects are novel. Moreover, the category of quasi orders with finite antichains is studied here for the first time, showing a natural example of category with terminal objects and equalisers, but lacking products.

Historically, well quasi orders have been defined as those quasi orders, i.e., sets equipped with a reflexive and transitive relation, which enjoy two properties: all the proper descending chains are finite, i.e., they are *well founded*, and all the antichains are finite. There are many alternative ways to characterise well quasi orders, see, e.g., Theorem 1. These alternative characterisations are equivalent assuming classical logic and a sufficiently strong set theory. In a constructive setting, these alternative definitions are more informative and, thus, widely used, see, e.g., [9].

In this paper, the original definition is retained since it seems to be more significant to analyse the categorical structure. In fact, descending chains and antichains interact in a quasi order, and requiring them to be finite induces some structure which is captured in the process of building limits and colimits; the counterexamples we will introduce, see Propositions 15 and 20, arise from a subtle interplay between finite and infinite, which the categorical construction makes explicit.

Exponentiation in the category of well quasi orders amounts to say when the space of monotone functions between two well quasi orders is a well quasi order. This is rarely the case, as it will be proved in Section 4. A similar result holds for well founded quasi orders and, in fact, exponential objects exist in the category of well quasi orders exactly when they exist in the category of well founded quasi orders.

The article starts by presenting the categorical structure of quasi orders in Section 2. Then, in Section 3, the subcategories of quasi orders with finite antichains, of well founded quasi orders, and of well quasi orders are defined and studied. Finally, Section 4 is devoted to characterise when exponential objects exist in the categories of well quasi orders and well founded quasi orders. The last section summarises the obtained results and hints at future developments.

2 Quasi orders in a categorical setting

Quasi orders are easily illustrated in categorical terms. Although their description is well known, see any introduction to category theory, e.g., [1] or [7], it is convenient to collect the fundamental results and proofs as they will be extensively used in the following. Also, since there are small differences in terminology and definitions in the literature, this section has the purpose to provide a non-ambiguous, clear reference for the results that will follow.

Definition 1 (Quasi orders and orders) A *quasi order* $\mathbb{Q} = \langle Q; \leq_Q \rangle$ is a set Q together with $\leq_Q \subseteq Q \times Q$, a reflexive and transitive relation. An *order* $\mathbb{O} = \langle O; \leq_O \rangle$ is a quasi order such that $\leq_O$ is anti-symmetric. To make notation uniform, we write $x < y$ for $x \leq y$ and $x \neq y$, $x \geq y$ for $y \leq x$, $x > y$ for $y < x$, $x \parallel y$ for $x \not\leq y$ and $y \not\leq x$, $x \sim y$ for $x \leq y$ and $y \leq x$.

Definition 2 (Monotone map) A *monotone map* or *(quasi order) homomorphism* is a function $f\colon P \to Q$ from the quasi order $\langle P; \leq_P \rangle$ to the quasi order $\langle Q; \leq_Q \rangle$ such that, for every pair $x, y \in P$, if $x \leq_P y$ then $f(x) \leq_Q f(y)$.

Definition 3 (QOrd and Ord) The category **QOrd** has the quasi orders as objects and their monotone maps as morphisms. The category **Ord** has the orders as objects and their homomorphisms as arrows.

Quasi orders and orders are represented as small categories through the functor $\mathrm{Rep}_{\mathbf{QOrd}}\colon \mathbf{QOrd} \to \mathbf{Cat}$ mapping $\langle Q; \leq_Q \rangle$ to the category $\mathbb{Q}$ having Q as the collection of objects such that $\mathrm{Hom}_{\mathbb{Q}}(x,y) = \emptyset$ when $x \not\leq_Q y$ and $\mathrm{Hom}_{\mathbb{Q}}(x,y) = \{x \leq y\}$, a singleton, when $x \leq_Q y$; if $f\colon \langle Q; \leq_Q \rangle \to \langle P; \leq_P \rangle$ is a monotone map, $\mathrm{Rep}_{\mathbf{QOrd}}(f)\colon \mathbb{Q} \to \mathbb{P}$ is defined as the functor mapping each $x \in \mathrm{Obj}\,\mathbb{Q}$ to $f(x) \in \mathrm{Obj}\,\mathbb{P}$, and each arrow $x \leq y\colon x \to y$ to $f(x) \leq f(y)\colon f(x) \to f(y)$.

Proposition 1 *Any small category such that* $|\mathrm{Hom}(x,y)| \leq 1$ *for any pair of objects* x, y*, can be interpreted as a quasi order. Moreover, if such a category is skeletal, that is, identities are the only isomorphisms, it can be interpreted as an order. Finally, any functor between quasi order categories as above is uniquely associated with a monotone map and vice versa.*

Depending on the context, we will use the algebraic presentation of Definition 1, or the categorical representation, indifferently.

Proposition 2 **Ord** *is a reflective subcategory of* **QOrd**.

Corollary 1 *Any limit or colimit which happens to exist in* **QOrd***, immediately exists in* **Ord***, too.*

Definition 4 (Forgetful functor) The *forgetful functor* $U_{\mathbf{QOrd}}\colon \mathbf{QOrd} \to \mathbf{Set}$ maps each object $\langle P; \leq_P \rangle$ to P and each arrow $f\colon \langle P; \leq_P \rangle \to \langle Q; \leq_Q \rangle$ to $f\colon P \to Q$.

Analogously, $U_{\mathbf{Ord}}\colon \mathbf{Ord} \to \mathbf{Set}$ is the subfunctor of $U_{\mathbf{QOrd}}$ restricted to **Ord**.

The algebraic view of quasi orders and orders, as in Definition 1, is, in fact, nothing else that looking at these structures through the forgetful functors. We usually describe these structures as sets equipped with a relation: the categorical description provides an alternative way of considering them, specifically, as categories (abstract structures) which become concrete (based on sets) when looked at with the appropriate glasses, the forgetful functors.

Proposition 3 *The functor* $U_{\mathbf{QOrd}}$ *has a left adjoint given by the map* $A \mapsto \langle A; = \rangle$*, and a right adjoint given by the map* $A \mapsto \langle A; A \times A \rangle$.

Corollary 2 *The left adjoint of* $U_{\mathbf{Ord}}$ *is given by the map* $A \mapsto \langle A; = \rangle$.

Since right adjoints preserve limits and left adjoints preserve colimits, see [7], we conclude that

Corollary 3 *The functors* $U_{\mathbf{QOrd}}$ *and* $U_{\mathbf{Ord}}$ *preserve all the limits in* **QOrd** *and* **Ord***, respectively. Moreover,* $U_{\mathbf{QOrd}}$ *preserves all the colimits in* **QOrd**.

It is worth remarking that the preservation property helps to shape limits and colimits in **QOrd**: the supporting set in the (co-)limit quasi order must be the corresponding (co-)limit in **Set**. We need to make explicit the constructions of the limits and the colimits which exist in **QOrd**, as these constructions play an important role in the theory of quasi orders.

Proposition 4 *The empty set with the only possible order, notation* **0***, is the initial object of* **QOrd**. *Moreover, any singleton set with the only possible order, notation* **1***, is a terminal object of* **QOrd**.

Proposition 5 *Let* $\langle P; \leq_P \rangle, \langle Q; \leq_Q \rangle \in \mathsf{Obj}\,\mathbf{QOrd}$. *Define*

$$\leq_{P\times Q} = \left\{ \left((p,q),(p',q')\right) : p \leq_P p' \text{ and } q \leq_Q q' \right\} \quad .$$

Then $\langle P \times Q; \leq_{P\times Q} \rangle$ *is the product*[1] *of* $\langle P; \leq_P \rangle$ *and* $\langle Q; \leq_Q \rangle$.

Proposition 6 *Let* $f, g \colon \langle P; \leq_P \rangle \to \langle Q; \leq_Q \rangle$ *in* **QOrd**. *Define*

$$E = \{x \in P \colon f(x) = g(x)\}$$

and $\leq_E$ *as the restriction of* $\leq_P$ *to* $E \times E$. *Then,* $\langle E; \leq_E \rangle$ *and the inclusion map* $i \colon E \to P$ *is the equaliser of* f *and* g *in* **QOrd**.

Since the category **QOrd** has a terminal object, binary products and equalisers, by Theorem 2.8.1 in [1] the following result holds:

Corollary 4 **QOrd** *is finitely complete.*

Proposition 7 *Let* $\langle P; \leq_P \rangle, \langle Q; \leq_Q \rangle \in \mathsf{Obj}\,\mathbf{QOrd}$. *Define* $\leq_{P \sqcup Q} = \leq_P \sqcup \leq_Q$. *Then* $\langle P \sqcup Q; \leq_{P\sqcup Q} \rangle$ *is the coproduct of* $\langle P; \leq_P \rangle$ *and* $\langle Q; \leq_Q \rangle$.

Proposition 8 *Let* $f, g \colon \langle P; \leq_P \rangle \to \langle Q; \leq_Q \rangle$ *in* **QOrd**. *Define* $E = Q/{\approx}$ *where* $\approx$ *is the minimal equivalence relation containing* $\{(f(x), g(x)) : x \in P\} \subseteq Q \times Q$. *Also, define* $e \colon Q \to E$ *as* $e(x) = [x]_\approx$ *for all* $x \in Q$. *Finally, define* $\leq_E$ *as the reflexive and transitive closure of* $\{([x]_\approx, [y]_\approx) : x \leq_Q y\}$. *Then* $\langle E; \leq_E \rangle$ *together with* e *is the coequaliser of* f *and* g *in* **QOrd**.

Since **QOrd** has an initial object, binary coproducts and coequalisers, dualising Corollary 4 we can conclude

Corollary 5 **QOrd** *is finitely cocomplete.*

Proposition 9 *Let* $\langle P; \leq_P \rangle$ *and* $\langle Q; \leq_Q \rangle$ *be quasi orders. Define*

$$\langle Q; \leq_Q \rangle^{\langle P; \leq_P \rangle} = \langle \{f \colon P \to Q \colon f \text{ is monotone}\}\,; \leq_{\mathsf{exp}} \rangle$$

where $f \leq_{\mathsf{exp}} g$ *if and only if* $f(x) \leq_Q g(x)$ *for all* $x \in P$. *Also, let*

$$ev \colon \langle Q; \leq_Q \rangle^{\langle P; \leq_P \rangle} \times \langle P; \leq_P \rangle \to \langle Q; \leq_Q \rangle$$

be the function defined by $ev(f, x) = f(x)$. *Then* $\langle Q; \leq_Q \rangle^{\langle P; \leq_P \rangle}$ *is the exponential object in* **QOrd** *with evaluation map* ev.

[1] Here and elsewhere, we denote the ordering as $\leq_{P\times Q}$: often this notation will be abbreviated to $\leq_\times$, or even to $\leq$, when there is no risk of ambiguity.

Thus, we can conclude

Corollary 6 **QOrd** *is Cartesian closed and finitely cocomplete.*

Corollary 7 **Ord** *is Cartesian closed and finitely cocomplete.*

Nevertheless, neither **QOrd** nor **Ord** are elementary toposes: in fact, they have no subobject classifier. To prove this fact it suffices to show a counterexample.

Firstly, $f\colon \langle P;\leq_P\rangle \to \langle Q;\leq_Q\rangle$ is mono exactly when f is injective. This fact is proved as in **Set**, see, e.g. Example 1.7.7.a in [1]. So, up to isomorphisms, it suffices to consider inclusion maps. Thus, when $i\colon \langle P;\leq_P\rangle \hookrightarrow \langle Q;\leq_Q\rangle$, it follows that $P \subseteq Q$ and $\leq_P \subseteq \leq_Q$ in **Set**, although $\leq_P$ may not be the restriction of $\leq_Q$ to $P \times P$.

Let α be the discrete order on two element a and b, and let β be the order on $\{a,b\}$ such that $a < b$. Now, there are $i_1\colon \alpha \hookrightarrow \beta$ and $i_2\colon \beta \hookrightarrow \beta$: as functions, $i_1(a) = i_2(a) = a$ and $i_1(b) = i_2(b) = b$, and i_1 and i_2 are evidently monotone. But α is not isomorphic to β, notation $\alpha \not\cong \beta$.

If **QOrd** or **Ord** have a subobject classifier, the following would be pullbacks

$$\begin{array}{ccc} \alpha & \overset{i_1}{\rightarrowtail} & \beta \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \chi_{i_1}} \\ \mathbf{1} & \underset{\text{true}}{\longrightarrow} & \Omega \end{array} \qquad \begin{array}{ccc} \beta & \overset{i_2}{\rightarrowtail} & \beta \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \chi_{i_2}} \\ \mathbf{1} & \underset{\text{true}}{\longrightarrow} & \Omega \end{array}$$

But, since $\text{true}\colon \mathbf{1} \to \Omega = \langle O;\leq_O\rangle$ identifies a unique element $\top \in \Omega$, it follows that $\chi_{i_1}(i_1(a)) = \top = \chi_{i_1}(i_1(b))$ and $\chi_{i_2}(i_2(a)) = \top = \chi_{i_2}(i_2(b))$. Thus the arrows $\chi_{i_1}\colon \beta \to O$ and $\chi_{i_2}\colon \beta \to O$ are the same in **Set**. Since $U_{\mathbf{QOrd}}$ and $U_{\mathbf{Ord}}$ preserve pullbacks, it means that χ_{i_1} and χ_{i_2} must be both monotone and equal in **QOrd**. But, α and β are both the pullback object of the same pair of arrows (true, χ), so $\alpha \cong \beta$ which is a contradiction. Thus no such Ω may exist in **QOrd** and **Ord**. In fact, α and β are well quasi orders, so both well-founded and with finite antichains. Hence, also the categories **WFAQOrd**, **WFQOrd**, and **AQOrd**, which will be introduced in the next section, do not have a subobject classifier.

3 Well quasi orders and related categories

Describing well quasi orders categorically has the double purpose to provide a precise and uniform presentation of their properties, and to clarify what is really needed to prove a result. Uniformity is important to show that some results are, in fact, instances of a general pattern.

We are particularly interested in two properties whose combination characterises well quasi orders: possessing finite antichains, and having finite proper descending chains. Thus, we will define the relevant notions, following the standard formulation, see, e.g., [2], and we define a subcategory of **QOrd**, containing the quasi orders satisfying the notion. Although most results are nothing more than exercises, a few of the following properties are, as far as the authors know, not immediate and not present in the current literature, specifically Proposition 15, quasi orders with finite antichains to do not have products, Propositions 20 and 21, well founded quasi orders do not have coequalisers while well quasi orders do.

Definition 5 (Antichain) Fixed $\langle P; \leq_p \rangle \in \mathrm{Obj}\,\mathbf{QOrd}$, an *antichain* is a set $S \subseteq P$ such that, for every pair $x, y \in S$, when $x \neq y$ neither $x \leq_P y$, nor $y \leq_P x$, a condition that will be summarised in the notation $x \parallel y$, x is *incomparable* with y.

Definition 6 (AQOrd and AOrd) The **AQOrd** category is the full subcategory of **QOrd** whose objects are the quasi orders whose antichains are all finite. Similarly **AOrd** is the full subcategory of **Ord** whose objects are the orders whose antichains are all finite. Evidently **AOrd** $\subseteq$ **AQOrd**.

Since the inclusions **AQOrd** $\hookrightarrow$ **QOrd** and **AOrd** $\hookrightarrow$ **Ord** do not have left adjoints, these subcategories are not reflective in **QOrd** and **Ord**, respectively.

Proposition 10 **AOrd** *is a reflective subcategory of* **AQOrd**.

Definition 7 (Descending chain) A *descending chain* in a quasi order $\langle P; \leq_P \rangle$ is a family $\{p_i\}_i$ of elements in P, indexed by an ordinal, such that $p_1 \geq_P p_2 \geq_P \cdots \geq_P p_n \geq_P \cdots$. A descending chain is said to be *proper* when, for each i, j with $i < j$ strictly, $p_i \not\leq_P p_j$, or, in other words, when there are no equivalent elements in it. A quasi order is said to be *well founded* when every proper descending chain is finite.

Definition 8 (WFQOrd and WFOrd) The **WFQOrd** category is the full subcategory of **QOrd** whose objects are the well founded quasi orders. Similarly **WFOrd** is the full subcategory of **Ord** whose objects are the well founded orders. Evidently **WFOrd** $\subseteq$ **WFQOrd**.

Again, it is clear that **WFQOrd** and **WFOrd** are not reflective in **QOrd** and **Ord**, respectively, since there is no canonical way to construct a well founded order out of an arbitrary order.

Proposition 11 **WFOrd** *is a reflective subcategory of* **WFQOrd**.

Definition 9 A quasi order $\langle P; \leq_P \rangle$ is a *well quasi order* when it is both well founded and it has finite antichains.

The definition of well quasi order as above is convenient for the present analysis since the relevant structural properties of well quasi orders arise from a subtle interplay between finite proper descending chains and finite antichains: for example, well founded quasi orders do not have coequalisers, while quasi order with finite antichains do. Then, but this fact is not immediately consequential, well quasi orders have coequalisers because they are, in particular, quasi orders with finite antichains. Nevertheless, well quasi orders could be characterised in a number of alternative ways, which are equivalent assuming classical logic and a sufficiently strong set theory, which means, usually, having the Axiom of Choice.

Theorem 1 *Fixed a quasi order* $\mathbb{P} = \langle P; \leq_P \rangle$*, the following are equivalent:*

1. $\mathbb{P}$ *is a well quasi order;*
2. *Any infinite sequence* $\{x_i\}_i$ *of elements in* $\mathbb{P}$ *contains an increasing pair:* $x_i \leq_P x_j$ *for some* $i < j$*;*

3. *Any infinite sequence* $\{x_i\}_i$ *of elements in* $\mathbb{P}$ *contains an infinite increasing subsequence:* $\left\{x_{n_j}\right\}_j$ *such that* $x_{n_i} \leq_P x_{n_j}$ *for every* $i < j$.

Proof This is a standard result in the theory of well quasi orders [6]. □

It is worth remarking that any descending chain of length ω in a well quasi order has a finite prefix followed by an infinite sequence of equivalent elements. This is a direct consequence of point (3) in Theorem 1.

From these equivalent ways to formulate what is a well quasi order, immediately follows that any ordinal is a well quasi order.

Definition 10 (WFAQOrd and WFAOrd) The **WFAQOrd** category is the full subcategory of **QOrd** whose objects are the well quasi orders. Similarly **WFAOrd** is the full subcategory of **Ord** whose objects are the well founded orders with finite antichains. Evidently **WFAOrd** $\subseteq$ **WFAQOrd**.

Evidently, **WFAQOrd** is not reflective in **QOrd**, **WFQOrd**, and **AQOrd**. The same holds for **WFAOrd** with respect to **Ord**, **WFOrd**, and **AOrd**. This fact comes from the impossibility to construct a canonical well founded (quasi) order from an arbitrary (quasi) order, even assuming finite antichains, and, similarly, to construct a canonical quasi order with finite antichains from an arbitrary (quasi) order, even assuming it to be well founded.

Proposition 12 **WFAOrd** *is a reflective subcategory of* **WFAQOrd**.

A reasonable question to ask is whether the set-theoretic presentation of the above defined quasi orders is strong enough to determine the shape of limits and colimits. It turns out that this is the case, even if not in a completely direct way.

Definition 11 (Forgetful functors) The forgetful functors $U_{\mathbf{AQOrd}}\colon$ **AQOrd** $\to$ **Set**, $U_{\mathbf{WFQOrd}}\colon$ **WFQOrd** $\to$ **Set**, and $U_{\mathbf{WFAQOrd}}\colon$ **WFAQOrd** $\to$ **Set** map each object $\langle P; \leq_P\rangle$ to P and each arrow $f\colon \langle P; \leq_P\rangle \to \langle Q; \leq_Q\rangle$ to $f\colon P \to Q$.

Also, the forgetful functors $U_{\mathbf{AOrd}}\colon$ **AOrd** $\to$ **Set**, $U_{\mathbf{WFOrd}}\colon$ **WFOrd** $\to$ **Set**, and $U_{\mathbf{WFAOrd}}\colon$ **WFAOrd** $\to$ **Set** are the subfunctors of $U_{\mathbf{AQOrd}}$, $U_{\mathbf{WFQOrd}}$, and $U_{\mathbf{WFAQOrd}}$ restricted to **AOrd**, **WFOrd**, and **WFAOrd**, respectively.

Since the canonical adjoints in **QOrd** are elementary, it follows that

Proposition 13 *The forgetful functors* $U_{\mathbf{AQOrd}}$, $U_{\mathbf{WFQOrd}}$, *and* $U_{\mathbf{WFAQOrd}}$ *have all a right adjoint, mapping* $S \mapsto \langle S; S \times S\rangle$. *Also, the forgetful functor* $U_{\mathbf{WFQOrd}}$ *has a left adjoint mapping* $S \mapsto \langle S; =\rangle$.

It is immediate to check that the above forgetful functors restricted to orders do not have right adjoints, and that only the forgetful functor $U_{\mathbf{WFOrd}}$ has a left adjoint given by $S \mapsto \langle S; =\rangle$. This happens because there is no canonical way to reconstruct an order with finite antichains from an infinite set, and because there is no canonical way to construct a maximal well order from an infinite set.

The structure of the categories introduced so far is simple, in most cases, but less regular than the one of **QOrd**.

Since any finite quasi order has necessarily finite antichains and finite descending chain, it follows that

Proposition 14 *The initial object in* **AQOrd**, **WFQOrd**, *and* **WFAQOrd** *is* **0**. *Similarly, the terminal object in* **AQOrd**, **WFQOrd**, *and* **WFAQOrd** *is* **1**.

It is clear that, when $A \times B$ in **QOrd** is in the same category $\mathbb{C}$ as A and B, with $\mathbb{C}$ one of **AQOrd**, **WFQOrd**, **WFAQOrd**, it is also the product of A and B in $\mathbb{C}$.

Proposition 15 *The categories* **AQOrd** *and* **AOrd** *do not have products.*

Proof Consider $\langle \mathbb{N}; \leq \rangle$, the usual order of natural numbers, and $\langle \{0,1\}^*; \leq_{\mathsf{lex}} \rangle$, the lexicographic order on the free monoid over $0 < 1$.

Both orders are total, thus with finite antichains, and, in $\langle \mathbb{N}; \leq \rangle$, $0 < 1 < 2 < \cdots$ is an infinite ascending chain, while, in $\langle \{0,1\}^*; \leq_{\mathsf{lex}} \rangle$, $1 > 01 > 001 > 0001 > \cdots$ is an infinite descending chain. Let

$$x_n = \underbrace{0 \cdots 0}_{n} 1$$

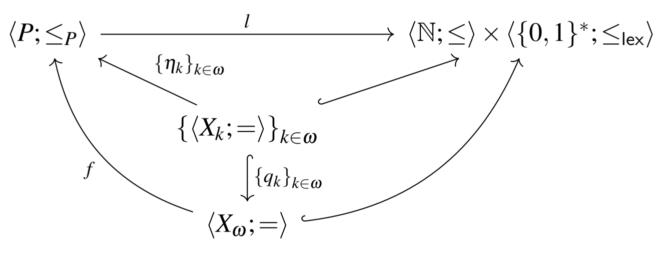
be the n-th term in the infinite descending chain above. Then $\{(n, x_n)\}_n$ is a sequence in $\langle \mathbb{N} \times \{0,1\}^*; \leq_\times \rangle$, the product order being as in Proposition 5. If $i \neq j$, then $(i, x_i) \parallel (j, x_j)$ since either $i < j$ and thus $x_i > x_j$, or $i > j$ and thus $x_i < x_j$, by definition. So $\{(n, x_n)\}_n$ is an infinite antichain. Hence, $\langle \mathbb{N}; \leq \rangle \times \langle \{0,1\}^*; \leq_{\mathsf{lex}} \rangle$ is not in **AQOrd**. However, at least in principle, **AQOrd** may have different products than **QOrd**.

But this is not the case. By contradiction, suppose there is the product $\langle P; \leq_P \rangle$ of $\langle \mathbb{N}; \leq \rangle$ and $\langle \{0,1\}^*; \leq_{\mathsf{lex}} \rangle$ in **AQOrd**. Let $X_k = \{(n, x_n) : n < k\}$ for $k \in \omega$, and $X_\omega = \{(n, x_n) : n < \omega\}$. Clearly, there are the obvious embeddings $q_k \colon X_k \hookrightarrow X_\omega$ and $X_k \hookrightarrow X_{k+1}$. Moreover, X_ω is the colimit of the diagram $X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_k \hookrightarrow \cdots$ in **Set**, and its canonical injections are the q_k arrows.

Since, for every $k \in \omega$, $(n, x_n) \mapsto n$ and $(n, x_n) \mapsto x_n$ are arrows $\langle X_k; = \rangle \to \langle \mathbb{N}; \leq \rangle$ and $\langle X_k; = \rangle \to \langle \{0,1\}^*; \leq_{\mathsf{lex}} \rangle$ in **AQOrd**, respectively, there is $\eta_k \colon X_k \to P$, the universal arrow of their product in **AQOrd**. Thus $(P, \{\eta_k\}_{k \in \omega})$ is a cocone over $X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_k \hookrightarrow \cdots$ in **Set**, hence there is $f \colon X_\omega \to P$, the universal arrow of the X_ω colimit. Also, $f \colon \langle X_\omega; = \rangle \to \langle P; \leq_P \rangle$ in **QOrd**.

Since $\langle \mathbb{N}; \leq \rangle \times \langle \{0,1\}^*; \leq_{\mathsf{lex}} \rangle$ exists in **QOrd**, there is $l \colon \langle P; \leq_P \rangle \to \langle \mathbb{N}; \leq \rangle \times \langle \{0,1\}^*; \leq_{\mathsf{lex}} \rangle$, the universal arrow of the $\times$ product. Moreover, the domain of the product $\langle \mathbb{N}; \leq \rangle \times \langle \{0,1\}^*; \leq_{\mathsf{lex}} \rangle$ is the vertex of a cocone over the diagram $X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_k \hookrightarrow \cdots$ in **Set**, with the obvious injections. Hence, there is the universal arrow of the X_ω colimit, which lifts to the inclusion $\langle X_\omega; = \rangle \hookrightarrow \langle \mathbb{N}; \leq \rangle \times \langle \{0,1\}^*; \leq_{\mathsf{lex}} \rangle$ in **QOrd**.

Putting all together, it follows that the diagram



commutes in **QOrd**; in particular, the top diagram commutes by the uniqueness of l. Consider $f(X_\omega)$: if $f(n,x_n) \leq_P f(m,x_m)$ with $n \neq m$, then $(l \circ f)(n,x_n) \leq_\times (l \circ f)(m,x_m)$ since l is monotone. But $l \circ f$ is an inclusion, so $(n,x_n) \leq_\times (m,x_m)$, contradicting X_ω to be an antichain. Then, $f(X_\omega)$ is an infinite antichain in $\langle P; \leq_P \rangle$ contradicting the assumption that $\langle P; \leq_P \rangle$ is in **AQOrd**. □

Actually, by the following proposition, the shown counterexample is general, in the sense that every counterexample arises from the combination of an infinite ascending chain and an infinite descending chain.

Proposition 16 *If* $\langle P; \leq_P \rangle$ *and* $\langle Q; \leq_Q \rangle$ *are both in* **WFQOrd** *or* **WFAQOrd**, *then* $\langle P \times Q; \leq_{P \times Q} \rangle$ *is their product.*

Proof Reminding Proposition 5, it suffices to show that $\langle P \times Q; \leq_{P \times Q} \rangle$ is in the right category when $\langle P; \leq_P \rangle$ and $\langle Q; \leq_Q \rangle$ are.

First, consider when $\langle P; \leq_P \rangle$ and $\langle Q; \leq_Q \rangle$ are both in **WFQOrd**. Let $\{(p_i, q_i)\}_i$ be a proper descending chain in $\langle P \times Q; \leq_\times \rangle$. Then, $\{p_i\}_i$ and $\{q_i\}_i$ are descending chains in $\langle P; \leq_P \rangle$ and $\langle Q; \leq_Q \rangle$, respectively. By compressing sequences of equivalent elements, they become proper chains, hence finite. Thus, by carefully tracing these chains in $\{(p_i, q_i)\}_i$, this must be finite, too.

When $\langle P; \leq_P \rangle$ and $\langle Q; \leq_Q \rangle$ are both in **WFAQOrd**, the result is already well-known in literature, and it can be shown to be valid by applying Dickson's lemma [3], or, alternatively, the infinite Ramsey theorem [5]. □

Proposition 17 *Let* $f, g \colon \langle P; \leq_P \rangle \to \langle Q; \leq_Q \rangle$ *be arrows in* **AQOrd**, **WFQOrd**, *or* **WFAQOrd**. *Define* $E = \{x \in P \colon f(x) = g(x)\}$, *and* $\leq_E$ *as the restriction of* $\leq_P$ *to* $E \times E$. *Then,* $\langle E; \leq_E \rangle$ *and the inclusion* $i \colon E \to P$ *is the equaliser of* f *and* g *in* **AQOrd**, **WFQOrd**, *and* **WFAQOrd**, *respectively.*

Corollary 8 **WFQOrd** *and* **WFAQOrd** *are finitely complete. Moreover,* **WFOrd** *and* **WFAOrd** *are finitely complete.*

Proposition 18 *If one of* **AQOrd**, **WFQOrd**, *or* **WFAQOrd** *contains both* $\langle P; \leq_P \rangle$ *and* $\langle Q; \leq_Q \rangle$, *then it also contains their coproduct* $\langle P \sqcup Q; \leq_{P \sqcup Q} \rangle$.

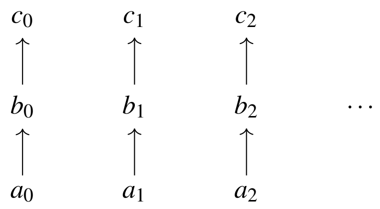
It is worth remarking that Proposition 18 can be immediately extended to arbitrary coproducts when all the quasi orders are in **WFQOrd**. Of course, the extension fails when dealing with finite antichains.

Proposition 19 **AQOrd** *has coequalisers.*

Proposition 20 **WFQOrd** *does not have coequalisers.*

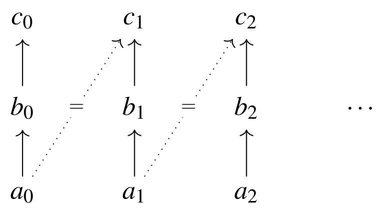
Proof Because of Proposition 13, and using the notation of Proposition 8, coequalisers in **WFQOrd**, when they exist, have the shape $\langle E; \leq_E \rangle$ with $E = Q/\approx$ and $\approx$ the minimal equivalence relation containing $\{(f(x), g(x)) : x \in P\}$. Since e must be monotone, $[x]_\approx \leq_E [y]_\approx$ when $x \leq_Q y$. Hence, $\leq_E$ must include the reflexive and transitive closure of $\{([x]_\approx, [y]_\approx) : x \leq_Q y\}$. By Proposition 8, $\langle E; \leq_E \rangle$ is a quasi order, so it remains to check whether it is well founded when $\langle P; \leq_P \rangle$ and $\langle Q; \leq_Q \rangle$ are.

Let $\langle \mathbb{N}; = \rangle$ be the set of natural numbers with the discrete order, which is well founded although it has infinite antichains, and let $\langle Q; \leq_Q \rangle$ be the disjoint union of ω copies of $a < b < c$:



Since every descending chain has length 3, $\langle Q; \leq_Q \rangle$ is well founded, although it has infinite antichains given by the sections in the picture above.

Let $f, g \colon \langle \mathbb{N}; = \rangle \to \langle Q; \leq_Q \rangle$ be defined as $f(n) = c_{n+1}$ and $g(n) = a_n$, respectively. Calculating, in the coequaliser one obtains



that is, the c_{n+1} element of $\langle Q; \leq_Q \rangle$ is identified with the a_n element in the quotient, for each $n \in \omega$. Thus, $\{b_i\}_{i \in \omega}$ is an infinite proper descending chain in $\langle E; \leq_E \rangle$. □

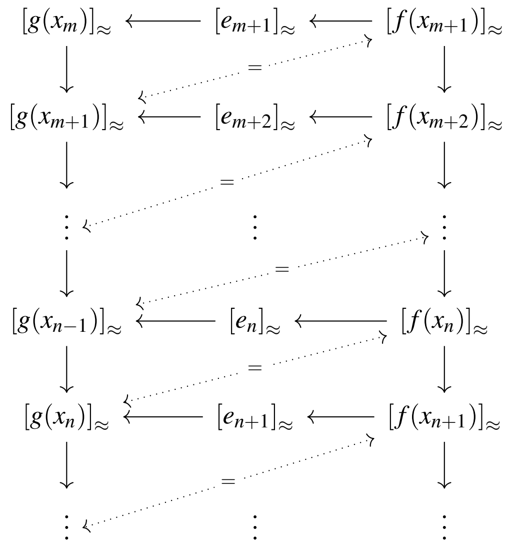
The above counterexample is general in the sense that every other possible counterexample arises from gluing together an infinite amount of finite descending chains along an infinite antichain. In fact, this is clear because of the following proposition, which shows that the same construction cannot be carried out in **WFAQOrd**, and this fact suffices to show that coequalisers exist.

Proposition 21 **WFAQOrd** *has coequalisers.*

Proof As in Proposition 20 and using the same notation, it suffices to show that the coequaliser object $\langle E; \leq_E \rangle$ is well founded, since it has the finite antichains property by Proposition 19. Let $\{[e_i]_\approx\}_{i \in I}$ be a descending chain in $\langle E; \leq_E \rangle$ with I an ordinal. Then $[e_0]_\approx \geq_E [e_1]_\approx \geq_E \cdots \geq_E [e_n]_\approx \geq_E \cdots$, thus there are appropriate elements in $\langle P; \leq_P \rangle$ such that either $e_0 \geq_Q f(x_0) \approx g(x_0) \geq_Q f(x_1) \approx \cdots \geq_Q f(x_k) \approx e_1 \geq_Q f(x_{k+1}) \approx g(x_{k+1}) \geq_Q \cdots$, or $e_0 \geq_Q g(x_0) \approx f(x_0) \geq_Q g(x_1) \approx \cdots \geq_Q g(x_k) \approx e_1 \geq_Q g(x_{k+1}) \approx f(x_{k+1}) \geq_Q \cdots$. In more compact terms, the sequence $\{e_i\}_i$ is definitely bounded by the sequences $\{f(x_i)\}_i$ and $\{g(x_i)\}_i$. For the sake of simplicity, we assume that g bounds e from above and f from below. Thus, it is completely general to assume that $\{e_i\}_{i \in I}$ is such that $g(x_n) \geq_Q e_{n+1} \geq_Q f(x_{n+1})$ for each $n \in I$, as we may always reduce to this condition.

We may safely restrict to $I \leq \omega$ since, if $I > \omega$, the prefix of length ω suffices to show whether the statement of the proposition holds, as well. Now, suppose $I = \omega$.

By Theorem 1, there is a subsequence of $\{x_i\}_{i\in\omega}$ which forms an infinite ascending chain. Moreover, as remarked after the statement of Theorem 1, its proof establishes that the subsequence has the form $\{x_i\}_{i\in\omega,i>m}$ for some $m \in \omega$. But f and g are both monotone, thus the sequences $\{f(x_i)\}_{i\in\omega,i>m}$ and $\{g(x_i)\}_{i\in\omega,i>m}$ are ascending chains in $\langle Q;\leq_Q\rangle$. Then, being the candidate coequaliser map e monotone, $\{[f(x_i)]_\approx\}_{i\in\omega,i>m}$ and $\{[g(x_i)]_\approx\}_{i\in\omega,i>m}$ are ascending chains in $\langle E;\leq_E\rangle$, thus



Then, for $n > m+1$, $[g(x_n)]_\approx \geq_E [g(x_{n-1})]_\approx \geq_E [e_n]_\approx \geq_E [f(x_n)]_\approx = [g(x_n)]_\approx$, and so $[e_n]_\approx$ is equivalent to $[g(x_n)]_\approx$ and to $[g(x_{n-1})]_\approx$. Thus, $[e_n]_\approx$ and $[e_{n+1}]_\approx$ are equivalent in $\langle E;\leq_E\rangle$. So, any descending chain of length ω in $\langle E;\leq_E\rangle$ is not proper, having an infinitely long tail of equivalent elements. □

The following proposition shows that the categories of interest in this section have no exponentiation.

Proposition 22 *The subcategories* **AQOrd**, **WFQOrd**, *and* **WFAQOrd** *do not have exponentiation.*

Proof Since **AQOrd** does not have products, it cannot have exponentiation.

Since $\langle\mathbb{N};\leq\rangle$ and $\mathbf{2} = \langle\{0,1\};\{(0,0),(0,1),(1,1)\}\rangle$ are both well orders, they lie in all the subcategories in the statement. Consider the family $\{f_i\colon \langle\mathbb{N};\leq\rangle \to \mathbf{2}\}_{i\in\omega}$, defined by

$$f_i(x) = \begin{cases} 1 & \text{if } x \geq i \\ 0 & \text{otherwise} \end{cases} .$$

Whenever $i < j$, $f_i(x) \geq f_j(x)$ for all $x \in \omega$, that is $f_i \geq f_j$, since

- if $x \geq j > i$, $f_i(x) = 1 = f_j(x)$;
- if $x < i < j$, $f_i(x) = 0 = f_j(x)$;
- if $i \leq x < j$, $f_i(x) = 1 > 0 = f_j(x)$.

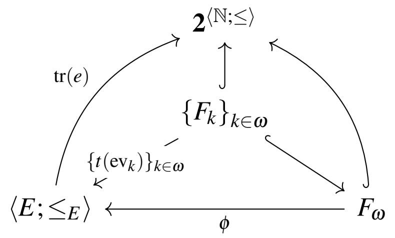
	0	1	$\times$	$+$	$\supset$	eq	coeq
QOrd	✓	✓	✓	✓	✓	✓	✓
AQOrd	✓	✓	-	✓	-	✓	✓
WFQOrd	✓	✓	✓	✓	-	✓	-
WFAQOrd	✓	✓	✓	✓	-	✓	✓

Table 1 Basic categorical structures in quasi orders.

But, when $i < j$, $f_i(i) = 1 > 0 = f_j(i)$, so $f_i > f_j$ strictly. Thus $f_0 > f_1 > f_2 > \cdots$ is an infinite proper descending chain, showing that $\mathbf{2}^{\langle \mathbb{N};\leq \rangle}$ is not well founded, thus it is not the exponential object in **WFQOrd** and **WFAQOrd**. However, in principle, another exponential object may exist.

By contradiction, suppose $\langle E;\leq_E \rangle$ is the exponential object representing the function space $\langle \mathbb{N};\leq \rangle \to \mathbf{2}$ in **WFQOrd** (or **WFAQOrd**), and let e be its evaluation arrow and t be its transpose function. Define $F_k = \langle \{f_i \colon i < k\}; \leq \rangle$, with $k \in \omega$ and $\leq$ the ordering on functions as in Proposition 9. Also, let $F_\omega = \langle \{f_i \colon i \in N\}; \leq \rangle$. Clearly, $F_k \hookrightarrow F_\omega$ and $F_k \hookrightarrow F_{k+1}$ for every $k \in \omega$. Call ev_k the restriction of the standard evaluation map ev in **QOrd** to $F_k \times \langle \mathbb{N} : \leq \rangle$.

It holds that F_ω is the colimit of the diagram $F_0 \hookrightarrow F_1 \hookrightarrow \cdots \hookrightarrow F_n \hookrightarrow \cdots$ in **QOrd** since injections are monotone, and the co-universal property follows because the co-universal arrow is uniquely defined by joining the values on the initial segments F_k of F_ω. But $\langle E;\leq_E \rangle$ is the vertex of a cocone on the same diagram, with $t(\mathrm{ev}_k)$ its injections, so there is an arrow $\phi \colon F_\omega \to \langle E;\leq_E \rangle$, being F_ω a colimit. Diagrammatically,



Since $F_\omega \hookrightarrow \mathbf{2}^{\langle \mathbb{N};\leq \rangle}$ is an inclusion, it is injective, and so ϕ is injective, too. Then $\{\phi(f_i)\}_{i\in\omega}$ not only is a descending chain in $\langle E;\leq_E \rangle$ because ϕ is monotone, but it is also proper because ϕ is injective. Hence, $\langle E;\leq_E \rangle$ cannot be well founded, contradicting what previously assumed. □

Summarising **WFAQOrd** has a very similar structure to **QOrd**, while **WFQOrd** does not have coequalisers, showing that quotients do not combine well with finite descending chains. The categorically irregular structure of limits in **AQOrd** shows a natural example of a category whose colimits are better behaved than limits. The basic categorical constructions in quasi orders are summarised in Table 1.

4 Exponentiation

In Section 3 it has been shown that the subcategories **AQOrd**, **WFQOrd**, and, in particular, **WFAQOrd** do not have exponentiation. In **AQOrd**, this follows because there are no products in general, and a counterexample has been shown that holds both in **WFQOrd** and **WFAQOrd**.

It makes no sense to characterise when exponential objects exist in **AQOrd** since, without a product, not only exponentiation cannot be defined, but there is no coherent sense in which it could have a universal property, as required. But in **WFQOrd** and **WFAQOrd** exponential objects may exist, even if not for arbitrary pairs of objects. So, to characterise when exponential objects $\mathscr{B}^{\mathscr{A}}$ exist in **WFAQOrd**, we distinguish three cases, which cover all the possibilities:

(a) when $\mathscr{B}$ is a well quasi order in which every pair of elements is either incomparable or equivalent;
(b) when $\mathscr{A}$ is a well quasi order in which there is at least one proper infinite ascending chain, and $\mathscr{B}$ is a well quasi order in which there is a pair of elements which are comparable and not equivalent;
(c) when $\mathscr{A}$ is a well quasi order in which every proper ascending chain is finite, and $\mathscr{B}$ is a well quasi order in which there is a pair of elements which are comparable and not equivalent.

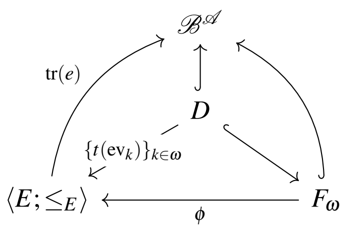
In some cases, although the exponential object $\mathscr{B}^{\mathscr{A}}$ exists in **QOrd**, it does not belong to **WFQOrd** or **WFAQOrd**. However, at least in principle, it could be possible that there is a *non-canonical* exponential object in those categories, i.e., one different from $\mathscr{B}^{\mathscr{A}}$. The following propositions rule out this possibility.

Proposition 23 *If $\mathscr{A},\mathscr{B} \in$ **WFQOrd** and $\mathscr{B}^{\mathscr{A}}$ contains a proper descending chain $\{f_i\}_{i\in\omega}$, then there is no exponential object from $\mathscr{A}$ to $\mathscr{B}$ in* **WFQOrd**.

Proof By contradiction, let $\langle E;\leq_E\rangle$, together with the evaluation map e and the exponential transpose function t, be the exponential object from $\mathscr{A}$ to $\mathscr{B}$ in **WFQOrd**.

Define $F_k = \langle\{f_i : i < k\};\leq\rangle$ for every $k \in \omega$, where the ordering is the usual pointwise ordering of functions. Also, define $F_\omega = \langle\{f_i : i \in \omega\};\leq\rangle$. Thus, there are the obvious injections $F_k \hookrightarrow F_{k+1}$ and $F_k \hookrightarrow F_\omega$ for each $k \in \omega$. Moreover, let D be the diagram $F_0 \hookrightarrow F_1 \hookrightarrow \cdots \hookrightarrow F_n \hookrightarrow \cdots$.

Clearly, F_ω is the colimit of D in **QOrd**. Calling tr and ev the transpose map and the evaluation map of the exponential object $\mathscr{B}^{\mathscr{A}}$ in **QOrd**, and calling ev_k the restriction of ev to F_k, the following diagram commutes



with ϕ the universal arrow of the F_ω colimit.

Since $\{f_i\}_{i\in\omega}$ is a proper descending chain, $\{\phi(f_i)\}_{i\in\omega}$ is a descending chain in $\langle E;\leq_E\rangle$. Suppose $\phi(f_i) \sim \phi(f_j)$ for some $i \neq j$. Then, being $\mathrm{tr}(e)$ monotone,

$\mathrm{tr}(e) \circ \phi(f_i) \sim \mathrm{tr}(e) \circ \phi(f_j)$, that is, $f_i \sim f_j$, contradicting the hypothesis in the statement. Hence $\{\phi(f_i)\}_{i\in\omega}$ is a proper descending chain in $\langle E; \leq_E\rangle$, contradicting the hypothesis that $\langle E; \leq_E\rangle$ is well founded. □

Corollary 9 *If $\mathscr{A}, \mathscr{B} \in$ **WFAQOrd** and $\mathscr{B}^{\mathscr{A}}$ contains a proper descending chain, then there is no exponential object from $\mathscr{A}$ to $\mathscr{B}$ in* **WFAQOrd**.

Proposition 24 *If $\mathscr{A}, \mathscr{B} \in$ **WFAQOrd** and $\mathscr{B}^{\mathscr{A}}$ contains $\{f_i\}_{i\in\omega}$, an infinite antichain, then there is no exponential object from $\mathscr{A}$ to $\mathscr{B}$ in* **WFAQOrd**.

Proof Following the same argument of Proposition 23, one derives $\mathrm{tr}(e) \circ \phi(f_i) = f_i$. If $\phi(f_i) \leq_E \phi(f_j)$ for some $i \neq j$ then, being $\mathrm{tr}(e)$ monotone, $f_i = \mathrm{tr}(e) \circ \phi(f_i) \leq \mathrm{tr}(e) \circ \phi(f_j) = f_j$, contradicting $f_i \parallel f_j$. Hence, $\{\phi(f_i)\}_{i\in\omega}$ is an infinite antichain in $\langle E; \leq_E\rangle$, which contradicts $\langle E; \leq_E\rangle$ to be in **WFAQOrd**. □

Reminding that $\sim$ is the relation identifying equivalent elements in a quasi order, the following properties are useful to construct the exponential objects.

Proposition 25 *Let $\mathscr{C}$ be a quasi order. Then $\mathscr{C}$ is a well quasi order if and only if $\mathscr{C}/\!\sim$ is a well quasi order.*

Proof $\mathscr{C}/\!\sim$ is the quasi order in which equivalent elements are identified, see Proposition 12. Every proper descending chain or antichain in $\mathscr{C}$ is so also in $\mathscr{C}/\!\sim$, and vice versa, since they do not contain equivalent elements. Thus, the conclusion follows immediately. □

Corollary 10 *Let $\mathscr{C}$ be a quasi order. Then $\mathscr{C}$ is a well founded quasi order if and only if $\mathscr{C}/\!\sim$ is so.*

Proposition 26 *In* **QOrd**, $\left(\mathscr{B}^{\mathscr{A}}\right)/\!\sim\, \cong (\mathscr{B}/\!\sim)^{\mathscr{A}}$.

Proof Notice how $f \sim g$ if and only if for each $x \in \mathscr{A}$, $f(x) \sim g(x)$. So if $[f]_\sim \in \left(\mathscr{B}^{\mathscr{A}}\right)/\!\sim$, being $f\colon \mathscr{A} \to \mathscr{B}$ monotone, $\bar{f}\colon \mathscr{A} \to \mathscr{B}/\!\sim$ such that $x \mapsto [f(x)]_\sim$ is well defined since, if $g \sim f$, then $g(x) \in [f(x)]_\sim$. Oppositely, given $f \in (\mathscr{B}/\!\sim)^{\mathscr{A}}$, let f' be a monotone function $\mathscr{A} \to \mathscr{B}$ such that $f'(x) \in f(x)$ for every $x \in \mathscr{A}$. The choice of $f'(x)$ does not matter since, if $f''(x)$ makes different choices, $f' \sim f''$. Thus the map $\alpha\colon f \mapsto [f']_\sim$ is well defined, and it inverts $\beta\colon g \mapsto \bar{g}$. □

Proposition 27 *If $\mathscr{A}$ and $\mathscr{B}$ are quasi orders, then $(\mathscr{B}/\!\sim)^{\mathscr{A}} \cong (\mathscr{B}/\!\sim)^{\mathscr{A}/\sim}$.*

Proof Let $\alpha\colon (\mathscr{B}/\!\sim)^{\mathscr{A}} \to (\mathscr{B}/\!\sim)^{\mathscr{A}/\sim}$ be the map $g \mapsto ([x]_\sim \mapsto g(x))$, which is well defined since, if $y \in [x]_\sim$, then $x \sim y$ and so $g(x) \sim g(y)$, that is $g(x) = g(y)$, because the codomain is a quotient by $\sim$. Conversely, let $\beta\colon (\mathscr{B}/\!\sim)^{\mathscr{A}/\sim} \to (\mathscr{B}/\!\sim)^{\mathscr{A}}$ be the function $f \mapsto (x \mapsto f([x]_\sim))$. Evidently, α and β are each one inverse. □

Analysing case (a) on page 13, let $\mathscr{A}$ be a well quasi order, and let $\mathscr{B}$ be a well quasi order such that, for all $x, y \in \mathscr{B}$, $x \sim y$ or $x \parallel y$. Let $\mathrel{\hat{=}}$ be the transitive closure of $\leq_A \cup \leq_A^{-1}$. Clearly, if $x \mathrel{\hat{=}} y$ then $f(x) \sim f(y)$ for every $f\colon \mathscr{A} \to \mathscr{B}$ monotone under the hypotheses of (a). Also, if $x \mathrel{\hat{\neq}} y$ then $x \parallel y$, although the converse does not hold.

Let $\{[x_i]\}_{i\in I}$ be the collection of all the equivalence classes of $\mathscr{A}/\doteq$. Since $[x_i] \neq [x_j]$ when $i \neq j$, $x_i \not\doteq x_j$, so $x_i \parallel x_j$. Hence $\{x_i\}_{i\in I}$ is an antichain in $\mathscr{A}$ and thus, by hypothesis, finite. So $\mathscr{A}/\doteq$ is finite.

Similarly, $\mathscr{B}/\sim$ is finite: otherwise $\mathscr{B}$ would contain an infinite antichain, contradicting the hypothesis that $\mathscr{B}$ is a well quasi order.

Proposition 28 *Under the hypotheses of case (a),* $(\mathscr{B}/\sim)^{\mathscr{A}} \cong (\mathscr{B}/\sim)^{\mathscr{A}/\doteq}$.

Proof Let $\alpha\colon (\mathscr{B}/\sim)^{\mathscr{A}} \to (\mathscr{B}/\sim)^{\mathscr{A}/\doteq}$ defined as $f \mapsto \left([x]_{\doteq} \mapsto [f(x)]_{\sim}\right)$. This map is well defined since, if $x \doteq y$, then $f(x) \sim f(y)$ as seen before. Consider the map $\beta\colon (\mathscr{B}/\sim)^{\mathscr{A}/\doteq} \to (\mathscr{B}/\sim)^{\mathscr{A}}$ given by $g \mapsto \left(x \mapsto g([x]_{\doteq})\right)$. Clearly, α and β are each other inverse. □

Summarising case (a), $\left(\mathscr{B}^{\mathscr{A}}\right)/\sim \cong (\mathscr{B}/\sim)^{\mathscr{A}} \cong (\mathscr{B}/\sim)^{\mathscr{A}/\doteq}$, and both $\mathscr{A}/\doteq$ and $\mathscr{B}/\sim$ are finite, thus $(\mathscr{B}/\sim)^{\mathscr{A}/\doteq}$ is finite and so is $(\mathscr{B})^{\mathscr{A}}/\sim$. By Proposition 25, $\mathscr{B}^{\mathscr{A}}$, which is a quasi order, is a well quasi order if and only if $\left(\mathscr{B}^{\mathscr{A}}\right)/\sim$ is, which is the case, being finite. In conclusion,

Lemma 1 *Under the hypotheses of case (a),* $\mathscr{B}^{\mathscr{A}}$ *is a well quasi order.*

Case (b) is simpler:

Proposition 29 *In case (b),* $\mathscr{B}^{\mathscr{A}}$ *contains an infinite proper descending chain.*

Proof We proceed generalising the counterexample in Proposition 22. Let $b_1 < b_2$ in $\mathscr{B}$ be a pair of comparable but not equivalent elements, and let $\{a_i\}_{i\in\omega}$ be an infinite proper ascending chain in $\mathscr{A}$. Define $\{f_i\colon \mathscr{A} \to \mathscr{B}\}_{i\in\omega}$ as

$$f_i(x) = \begin{cases} b_2 & \text{if } a_i \le x \\ b_1 & \text{otherwise} \end{cases}.$$

Clearly, when $x < y$, if $a_i \le x$, then $f_i(x) = f_i(y)$, otherwise, $x < a_i$ or $x \parallel a_i$, and thus $f_i(x) = b_1 \le f_i(y)$. So, for every $i \in \omega$, f_i is monotone.

Moreover, when $i < j$,

- if $a_i < a_j \le x$, then $f_i(x) = b_2 = f_j(x)$, so $f_j(x) \le f_i(x)$;
- if $x < a_i < a_j$, then $f_i(x) = b_1 = f_j(x)$, so $f_j(x) \le f_i(x)$;
- if $a_i \le x < a_j$, then $f_i(x) = b_2 > b_1 = f_j(x)$, so $f_j(x) \le f_i(x)$;
- if $x \parallel a_i$ and $x \parallel a_j$, $f_i(x) = b_1 = f_j(x)$, so $f_j(x) \le f_i(x)$;
- if $x \parallel a_i$ and $x < a_j$, $f_i(x) = b_1 = f_j(x)$, so $f_j(x) \le f_i(x)$;
- if $x \parallel a_j$ and $a_i \le x$, $f_i(x) = b_2 > b_1 = f_j(x)$, so $f_j(x) \le f_i(x)$.

Since $a_i < a_j$, it cannot happen that $x \parallel a_i$ and $a_j \le x$, or $x \parallel a_j$ and $x \le a_i$. So, for every $x \in A$, $f_j(x) \le f_i(x)$, that is, $f_j \le f_i$ when $i < j$. Moreover, since $f_i(a_i) = b_2 \neq f_j(a_i)$ when $i < j$, $f_i < f_j$ strictly. Thus, $\{f_i\}_{i\in\omega}$ is a proper descending chain in $\mathscr{B}^{\mathscr{A}}$. □

Lemma 2 *In case (b), there is no exponential object representing the function space from* $\mathscr{A}$ *to* $\mathscr{B}$.

Proof Immediate, by Proposition 29 and Corollary 9. □

Lemma 3 *Under the hypotheses of case (c), $\mathscr{B}^{\mathscr{A}}$ is a well quasi order.*

Proof Let $\mathscr{B}$ be a well quasi order in which there are $x < y$, and let $\mathscr{A}$ be a well quasi order such that every proper ascending chain is finite. Suppose $\mathscr{A}/{\sim}$ to be infinite. Then, by the Axiom of Choice applied to $\mathscr{A}/{\sim}$, there is $\{x_i\}_{i\in\omega}$ in $\mathscr{A}$ such that all the elements x_i are non equivalent. Thus, by Theorem 1, there is $\{y_i\}_{i\in\omega} \subseteq \{x_i\}_{i\in\omega}$ such that $y_i < y_j$ for $i < j$. Hence $\{y_i\}_{i\in\omega}$ is an infinite proper ascending chain in $\mathscr{A}$ contradicting the hypothesis. So, $\mathscr{A}/{\sim}$ must be finite, and thus $\mathscr{A}$ contains just a finite amount of non equivalent elements.

Let $n = |\mathscr{A}/{\sim}|$, and assume $\{x_i\}_{i<n}$ to be a total well ordering of $\mathscr{A}/{\sim}$. Let $f \in (\mathscr{B}/{\sim})^{\mathscr{A}/\sim}$, i.e., $f\colon \mathscr{A}/{\sim} \to \mathscr{B}/{\sim}$ monotone with respect to[2] $\leq_{\mathsf{exp}}$. Then $(f(x_0),\ldots, f(x_{n-1})) \in (\mathscr{B}/{\sim})^n$, and so there is an injection $(\mathscr{B}/{\sim})^{\mathscr{A}/\sim} \to (\mathscr{B}/{\sim})^n$.

By Proposition 25 and the hypotheses, $\mathscr{B}/{\sim}$ is a well quasi order, so by iterating Dickson's Lemma n times $(\mathscr{B}/{\sim})^n$ is a well quasi order with respect to component-wise ordering. Thus $\langle(\mathscr{B}/{\sim})^{\mathscr{A}/\sim}; \leq'_{\mathsf{exp}}\rangle$, with $\leq'_{\mathsf{exp}}$ identifying equivalent elements and otherwise acting as $\leq_{\mathsf{exp}}$, is a well quasi order because $\leq_{\mathsf{exp}}$ corresponds to component-wise ordering via the injection above, and the image of $(\mathscr{B}/{\sim})^{\mathscr{A}/\sim}$ is thus a well quasi order, being a suborder of $(\mathscr{B}/{\sim})^n$, and, moreover, it is isomorphic to $(\mathscr{B}/{\sim})^{\mathscr{A}/\sim}$.

Hence, $(\mathscr{B}/{\sim})^{\mathscr{A}/\sim} \cong (\mathscr{B}/{\sim})^{\mathscr{A}}$ by Proposition 27 and so, by Proposition 26, $(\mathscr{B}/{\sim})^{\mathscr{A}/\sim} \cong \left(\mathscr{B}^{\mathscr{A}}\right)/{\sim}$. Thus, as they share the same ordering relation $\leq'_{\mathsf{exp}}$, it follows that $\left(\mathscr{B}^{\mathscr{A}}\right)/{\sim}$ is a well quasi order. Thus, by Proposition 25, $\mathscr{B}^{\mathscr{A}}$ is a well quasi order under the hypotheses of case (c). □

Hence, the characterisation of exponential objects can be summarised as follows:

Theorem 2 (Exponential characterisation of WFAQOrd) *Let $\mathscr{A}$ and $\mathscr{B}$ be well quasi orders. Then, the exponential object $\mathscr{B}^{\mathscr{A}}$ exists in* **WFAQOrd** *and coincides with the one in* **QOrd** *if and only if either for every x,y in $\mathscr{B}$, $x \not< y$ strictly, or every proper ascending chain in $\mathscr{A}$ is finite.*

It is worth noticing that the counterexample of Proposition 22 is general, in the sense that each time the exponential object does not lie in **WFAQOrd**, it contains an infinite proper descending chain constructed as a sequence of functions dominating each other along an infinite ascending chain, see case (b).

Moving to the category **WFQOrd** of well founded quasi orders, Lemmata 2 and 3 hold even when $\mathscr{A}$ and $\mathscr{B}$ are just well founded quasi orders. Lemma 1 does not hold when $\mathscr{B}$ is a well founded quasi order which is not a well quasi order, because $\mathscr{B}/\sim$ is not always finite. But an analogous result can be shown:

Lemma 4 *Under the hypothesis of case (a), $\mathscr{B}^{\mathscr{A}}$ is a well founded quasi order.*

Proof Let $\{f_i\}_{i\in\omega}$ be an infinite descending chain in $\mathscr{B}^{\mathscr{A}}$, not necessarily proper. If $f_{i+1} \leq f_i$ then, by definition, for all $x \in \mathscr{A}$, $f_{i+1}(x) \leq_B f_i(x)$. But, in $\mathscr{B}$, every pair of elements which are comparable are also equivalent, then for all $x \in \mathscr{A}$, $f_{i+1}(x) \sim f_i(x)$. So $f_{i+1} \sim f_i$. Thus, every infinite descending chain is not proper and $\mathscr{B}^{\mathscr{A}}$ is a well founded quasi order. □

[2] Here, distinguishing the exponential ordering matters, so we use the subscript.

Moreover, for well founded quasi orders cases (a), (b) and (c) are not exhaustive. Indeed, we have also

(d) when $\mathscr{A}$ is a well founded quasi order in which there is at least one infinite antichain, and $\mathscr{B}$ is a well founded quasi order in which there is a pair of elements which are comparable and not equivalent.

And it holds that

Lemma 5 *In case (d), $\mathscr{B}^{\mathscr{A}}$ contains an infinite proper descending chain.*

Proof Let $b_1 < b_2$ in $\mathscr{B}$ be a pair of comparable but not equivalent elements, and let $\{a_i\}_{i\in\omega}$ be an infinite antichain in $\mathscr{A}$. Define $\{f_i \colon \mathscr{A} \to \mathscr{B}\}_{i\in\omega}$ as

$$f_i(x) = \begin{cases} b_2 & \text{if } a_j \leq x \text{ for some } j \geq i \\ b_1 & \text{otherwise} \end{cases} .$$

Let $x \leq y$. If $a_j \leq x \leq y$ for some $j \geq i$, $f_i(x) = b_2 \leq b_2 = f_i(y)$; otherwise $x < a_j$ or $x \parallel a_j$ for every $j \geq i$, and $f_i(x) = b_1 \leq f_i(y)$. Thus, f_i is monotone for every $i \in \omega$.

When $i \leq j$, for any $x \in A$

- if there is an index $k \geq i$ such that $a_k \leq x$, $f_i(x) = b_2 \geq f_j(x)$;
- otherwise $f_i(x) = b_1 \geq b_1 = f_j(x)$.

Thus, for any $x \in A$ $f_i(x) \geq f_j(x)$, that is, $f_i \geq f_j$. Moreover, if $i < j$, $f_i > f_j$; since $a_i \parallel a_k$ for any $k \neq i$, hence $f_i(a_i) = b_2 > b_1 = f_j(a_i)$. So $\{f_i\}_{i\in\omega}$ is an infinite proper descending chain in $\mathscr{B}^{\mathscr{A}}$. □

Thus, by Proposition 23, we can characterise exponential objects in **WFQOrd**:

Theorem 3 (Exponential characterisation of WFQOrd) *When $\mathscr{A}$ and $\mathscr{B}$ are well founded quasi orders, the exponential object $\mathscr{B}^{\mathscr{A}}$ exists in* **WFQOrd** *and coincides with the one in* **QOrd** *if and only if either for every x, y in $\mathscr{B}$, $x \not< y$ strictly, or every proper ascending chain and every antichain in $\mathscr{A}$ are finite.*

Recalling Theorem 2 we notice that, if $\mathscr{A}$ and $\mathscr{B}$ are both well quasi orders, then $\mathscr{B}^{\mathscr{A}}$ is in **WFAQOrd** exactly when it exists in **WFQOrd**.

4.1 Higman's Lemma

The key construction leading to rule out candidates for exponentiation in Propositions 23 and 24, which has also been used to show that **AQOrd** does not have products, see Proposition 15, encodes the idea of constructing an object by approximating it. Although this technique alone does not allow to prove Higman's Lemma, it is instructive to see how it neatly permits to state the Lemma in categorical terms. To remind, fixed a quasi order $\langle A; \leq_A\rangle$, the quasi order $\langle A^*; \leq_*\rangle$ is defined by posing A^* to be the set of finite sequences over A, and $[x_1, \dots, x_n] \leq_* [y_1, \dots, y_m]$ if and only if there is a strictly monotone function $f \colon \{1, \dots, n\} \to \{1, \dots, m\}$ such that $x_i \leq_A y_{f(i)}$ for each $1 \leq i \leq n$. Then, Higman's Lemma states

Lemma 6 **(Higman)** *$\mathscr{H} = \langle A^*; \leq_* \rangle$ is a well quasi order if and only if $\langle A; \leq_A \rangle$ is so.*

Let $n \in \omega$ and consider $\langle n; = \rangle$, the discrete well quasi order over n, and fix $\langle A; \leq_A \rangle$ to be a well quasi order. Then, by Theorem 2, $\langle A; \leq_A \rangle^{\langle n; = \rangle}$ is a well quasi order. It will be denoted as A^n in the following, and this notation is not ambiguous. In fact, let f be in A^n, so $f \colon \langle n; = \rangle \to \langle A; \leq_A \rangle$ and, thus, $f \cong \{f(i)\}_{i<n}$, by tabulating the function. Hence, A^n is also the collection of sequences of length n over A. Moreover, since $f \leq_{\mathsf{exp}} g$ is equivalent to $f(i) \leq_A g(i)$ for all $i < n$, that is, $\{f(i)\}_{i<n} \leq_* \{g(i)\}_{i<n}$, $\leq_{\mathsf{exp}}$ coincides with $\leq_*$ in A^n. Also $(f(0), \dots, f(n-1)) \leq_\times (g(0), \dots, g(n-1))$ in the Cartesian product A^n, see Dickson's Lemma, i.e., Proposition 16, so $\leq_{\mathsf{exp}} = \leq_* = \leq_\times$ in A^n. Thus, A^n can be equivalently interpreted as a function space, a collection of sequences, and a collection of vectors.

Define $A^{\leq n}$, with $n \in \omega$ as the quasi order category having as the collection of objects $\bigcup_{i \leq n} A^i$, and such that $\mathrm{Hom}_{A^{\leq n}}(f, g)$ is non empty exactly when $f \leq_* g$. It is clear that the sequence of orders $\{A^{\leq n}\}_{n \in \omega}$ approximates $\mathscr{H}$: as sets, the $A^{\leq n}$'s contain the finite sequences over A whose length is limited to n at most; as orders, the ordering $\leq_*$ on $A^{\leq n}$ is a suborder of $\leq_*$ on A^*, i.e., $A^{\leq n}$ is a full subcategory of $\mathscr{H}$. The sequence of the full subcategories $A^{\leq n}$ tends to cover the whole $\mathscr{H}$. What Higman's Lemma says is that this approximation preserves the property of being well quasi ordered. In fact

Proposition 30 *For every $n \in \omega$, $A^{\leq n}$ is a well quasi order.*

Proof By Theorem 2, A^i with $i \leq n$ is a well quasi order. So, by Proposition 18 iterated over i, $\bigsqcup_{i \leq n} A^i$ is a well quasi order. But $\bigsqcup_{i \leq n} A^i$ is a suborder of $A^{\leq n}$ on the whole $\bigcup_{i \leq n} A^i$, so $A^{\leq n}$ is a well quasi order too. □

Define $\mathscr{D}_\omega$ as the diagram in **QOrd** composed by all the objects $A^{\leq n}$, with $n \in \omega$, and their inclusions $A^{\leq n} \hookrightarrow A^{\leq m}$ when $n \leq m$. That is, $\mathscr{D}_\omega$ is the sequence of approximating suborders, each one containing the previous ones, along the inclusions. Clearly, the *limit* of the sequence is the minimal object containing all the suborders, via their injections, so it is, properly speaking, a colimit. Then, Higman's Lemma can be stated as

Lemma 7 **(Higman)** *The colimit of the diagram $\mathscr{D}_\omega$ exists in* **WFAQOrd**.

In fact, this statement is equivalent to the standard Lemma 6. Suppose $\langle H; \leq_H \rangle$ is the colimit of $\mathscr{D}_\omega$ in **QOrd**. Since the forgetful functor $U_{\mathbf{QOrd}}$ preserves colimits, see Corollary 3, it is clear that $U_{\mathbf{QOrd}}\left(A^{\leq n}\right) = \bigcup_{i \leq n} A^i$, so $U_{\mathbf{QOrd}}(\langle H; \leq_H \rangle) = \bigcup_{n \in \omega} \bigcup_{i \leq n} A^i = \bigcup_{i \in \omega} A^i = A^*$. Hence $H = A^*$, that is, the domain of the sought colimit is uniquely defined by the forgetful functor.

Let $f \in A^n$, $g \in A^m$ and $f \leq_* g$; then $n \leq m$, $f \in A^{\leq m}$, $g \in A^{\leq m}$ and $f \leq_* g$, so it follows that $\mathrm{Hom}_{A^{\leq m}}(f, g) \neq \emptyset$. Since the limit cocone injection $\langle A^{\leq m}; \leq_* \rangle \to \langle H; \leq_H \rangle$ becomes the inclusion $A^{\leq m} \hookrightarrow H$ when the forgetful functor $U_{\mathbf{QOrd}}$ is applied, the injection $\langle A^{\leq m}; \leq_* \rangle \to \langle H; \leq_H \rangle$ is identified as a function. However, this means that the arrow $f \to g$ in $A^{\leq m}$ becomes an arrow in $\langle H; \leq_H \rangle$, that is $f \leq_H g$. Conversely, if $f \leq_H g$, then, by co-universality of the colimit, there is $m \in \omega$ such that $f, g \in A^{\leq m}$ and $f \leq_* g$. Hence $\langle H; \leq_H \rangle = \langle A^*; \leq_* \rangle = \mathscr{H}$, and Lemma 7 states that this structure is a well quasi order.

5 Conclusions

The article illustrates the shape of finite limits and finite colimits in the categories of quasi orders, well founded quasi orders, quasi orders with finite antichains, and well quasi orders. The obtained results are summarised in Table 1. The novel contribution in this respect are the results about coequalisers, which have not been explored before. Also, the category of quasi orders with finite antichains has not been systematically studied till now, and it provides a natural example of category with all the finite colimits failing to be finitely complete, since it has a terminal object and equalisers, but it lacks products.

Going beyond limits and colimits, exponential objects have been considered in these categories. Although only the category of quasi orders has exponentiation, Theorems 2 and 3 characterise exactly when exponential objects exist in the categories of well and well founded quasi orders, respectively. These results are new.

All together these results provide a systematic view of the fundamentals of the theory of well quasi orders, in terms of category theory. Thus, the main contribution of the article lies in a complete characterisation of the fundamental structural properties of quasi orders, well founded quasi orders, quasi orders with finite antichains, and well quasi orders.

The first and obvious step in future research will be to analyse better quasi orders along the same lines. This is not immediate, since better quasi orders do not have a 'naturally categorical' structure in the framework which has been illustrated here. Also, quasi orders together with open maps are objects of interest, worth analysing.

Also, the results in this paper are strictly classical, depending on the Excluded Middle Principle and the Axiom of Choice. A constructive development of the presented results is possible and advisable, which is an objective of our future research. In this respect, among the many others, it is worth citing [9], to see how a piece of the theory of well quasi orders can and should be made constructive.

Acknowledgements This work has been supported by the project *Correctness by Construction* (CORCON), EU 7th framework programme, grant n. PIRSES-GA-2013-612638, European Union Research Agency, and by the project *Abstract Mathematics for Actual Computation: Hilbert's Program in the 21st Century*, John Templeton Foundation.

References

1. Borceux, F.: Handbook of Categorical Algebra 1: Basic Category Theory, *Encyclopedia of Mathematics and Its Applications*, vol. 50. Cambridge University Press (1994)
2. Davey, B.A., Priestley, H.A.: Introduction to Lattices and Order, second edn. Cambridge University Press (2002)
3. Dickson, L.E.: Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors. American Journal of Mathematics **35**(4), 413–422 (1913)
4. Gallier, J.H.: What's so special about Kruskal's theorem and the ordinal Γ_o? A survey of some results in proof theory. Annals of Pure and Applied Logic **53**(3), 199âĂŞ–260 (1991)
5. Graham, R.L., Rothschild, B.L., Spencer, J.H.: Ramsey Theory. John Wiley & Sons (1990)
6. Kruskal, J.B.: The theory of well-quasi-ordering: A frequently discovered concept. Journal of Combinatorial Theory (A) **13**, 297–305 (1972)
7. Mac Lane, S.: Categories for the Working Mathematician, *Graduate Texts in Mathematics*, vol. 5, second edn. Springer-Verlag (1997)

8. Milner, E.C.: Basic WQO- and BQO-theory. In: I. Rival (ed.) Graphs and Order, *NATO ASI Series*, vol. 147, pp. 487–502. Springer-Verlag (1985)
9. Seisenberger, M.: An inductive version of Nash-Williams' minimal-bad-sequence argument for Higman's lemma. In: P. Callaghan, Z. Luo, J. McKinna, R. Pollack (eds.) Types for Proofs and Programs (TYPES'00), *Lecture Notes in Computer Science*, vol. 2277, pp. 233–242. Springer-Verlag (2001)

A Proofs

This section is not intended for publication: it is provided for the reviewers in order to check the results in the paper.

Proposition 2, page 3: **Ord** *is a reflective subcategory of* **QOrd**.

Proof The inclusion functor $i\colon \mathbf{Ord} \to \mathbf{QOrd}$ is necessarily full and faithful, and **Ord** is obviously replete [1, p. 188]. So, it remains to show that i has a left adjoint. Let $\langle P; \leq_P \rangle \in \operatorname{Obj} \mathbf{QOrd}$: define $R = P/\!\sim$ where $\sim\, \subseteq P \times P$ is such that $p \sim q$ if and only if $p \leq_P q$ and $q \leq_P p$. Evidently, $\sim$ is an equivalence relation. Also, define $\leq_R \subseteq R \times R$ as $[p]_\sim \leq_R [q]_\sim$ if and only if $p \leq_P q$. It is immediate to check that $\langle R; \leq_R \rangle \in \operatorname{Obj} \mathbf{Ord}$. By unfolding the definition, the functor $\mathscr{R}\colon \mathbf{QOrd} \to \mathbf{Ord}$, mapping $\langle P, \leq_P \rangle$ to $\langle R, \leq_R \rangle$ as above, is a left adjoint to i with the unit $\eta_{\langle P, \leq_P \rangle}(x) = [x]_\sim$. □

Corollary 1, page 3: *Any limit or colimit which happens to exist in* **QOrd***, immediately exists in* **Ord***, too.*

Proof See [1, Section 3.5]. □

Proposition 4, page 4: *The empty set with the only possible order, notation* **0***, is the initial object of* **QOrd**. *Moreover, any singleton set with the only possible order, notation* **1***, is a terminal object of* **QOrd**.

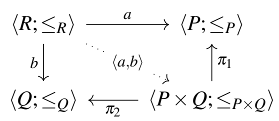
Proof Since $\langle \emptyset, \emptyset \rangle$ is trivially a quasi order, it suffices to notice that $f\colon \emptyset \to S$ is uniquely defined and monotone for any set S. Also, since $\langle \{\bullet\}, \{(\bullet,\bullet)\} \rangle$ is a quasi order, it suffices to notice that $g\colon S \to \{\bullet\}$ is uniquely defined and monotone. □

Proposition 5, page 4: *Let* $\langle P; \leq_P \rangle, \langle Q; \leq_Q \rangle \in \operatorname{Obj} \mathbf{QOrd}$. *Define*

$$\leq_{P \times Q} = \left\{ \left((p,q),(p',q')\right) : p \leq_P p' \text{ and } q \leq_Q q' \right\} \quad .$$

Then $\langle P \times Q; \leq_{P \times Q} \rangle$ *is the product of* $\langle P; \leq_P \rangle$ *and* $\langle Q; \leq_Q \rangle$.

Proof It is immediate to see that $\langle P \times Q; \leq_{P \times Q} \rangle$ is a quasi order. Also, the projections $\pi_1 : P \times Q \to P$ and $\pi_2 : P \times Q \to Q$ are monotone. Consider the diagram in **QOrd**:



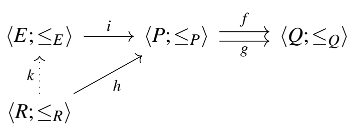
Define $\langle a,b \rangle(x) = (a(x), b(x))$ for all $x \in R$. If $x \leq_R y$ then $\langle a,b \rangle(x) \leq_{P \times Q} \langle a,b \rangle(y)$ since $a(x) \leq_P a(y)$ and $b(x) \leq_Q b(y)$. Let $f\colon \langle R; \leq_R \rangle \to \langle P \times Q; \leq_{P \times Q} \rangle$ be such that $\pi_1 \circ f = a$ and $\pi_2 \circ f = b$. Then $\pi_1 \circ f = \pi_1 \circ \langle a,b \rangle$ and $\pi_2 \circ f = \pi_2 \circ \langle a,b \rangle$. So, since $f(x) = (\alpha, \beta)$, $\alpha = (\pi_1 \circ f)(x) = (\pi_1 \circ \langle a,b \rangle)(x) = a(x)$ and $\beta = (\pi_2 \circ f)(x) = (\pi_2 \circ \langle a,b \rangle)(x) = b(x)$, that is, $f(x) = (a(x), b(x)) = \langle a,b \rangle(x)$. □

Proposition 6, page 4: *Let* $f, g\colon \langle P; \leq_P \rangle \to \langle Q; \leq_Q \rangle$ *in* **QOrd**. *Define*

$$E = \{x \in P \colon f(x) = g(x)\}$$

and $\leq_E$ *as the restriction of* $\leq_P$ *to* $E \times E$. *Then,* $\langle E; \leq_E \rangle$ *and the inclusion map* $i\colon E \to P$ *is the equaliser of* f *and* g *in* **QOrd**.

Proof It is immediate to see that $\langle E; \leq_E \rangle$ is a quasi order and that the inclusion map is monotone. By definition of E, $f \circ i = g \circ i$. Thus, consider the diagram in **QOrd**:

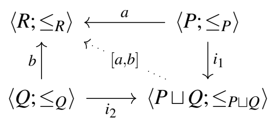


Let $\langle R; \leq_R \rangle \in \operatorname{Obj} \mathbf{QOrd}$ and $h\colon \langle R; \leq_R \rangle \to \langle P; \leq_P \rangle$ be such that $f \circ h = g \circ h$. Define in **Set**, $k(x) = h(x)$ for all $x \in R$, thus $k\colon R \to E$ since $(f \circ h)(x) = (g \circ h)(x)$ implies that $f(h(x)) = g(h(x))$, so $h(x) = k(x) \in E$ for all $x \in R$.

Evidently, k is monotone since h is, and $i \circ k = h$ by definition. Being i an inclusion, k is also necessarily unique. □

Proposition 7, page 4: *Let $\langle P; \leq_P \rangle, \langle Q; \leq_Q \rangle \in \operatorname{Obj} \mathbf{QOrd}$. Define $\leq_{P \sqcup Q} = \leq_P \sqcup \leq_Q$. Then $\langle P \sqcup Q; \leq_{P \sqcup Q} \rangle$ is the coproduct of $\langle P; \leq_P \rangle$ and $\langle Q; \leq_Q \rangle$.*

Proof It is immediate to see that $\langle P \sqcup Q; \leq_{P \sqcup Q} \rangle$ is a quasi order, and the injections $i_1 \colon P \to P \sqcup Q$ and $i_2 \colon Q \to P \sqcup Q$ are monotone. Thus, consider the following diagram in **QOrd**:



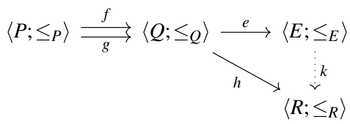
Define

$$[a,b](x) = \begin{cases} a(x) & \text{if } x \in P \\ b(x) & \text{if } x \in Q \end{cases} ,$$

If $x \leq_{P \sqcup Q} y$ then either $x \leq_P y$ or $x \leq_Q y$, so either $[a,b](x) = a(x) \leq_P a(y) = [a,b](y)$, or $[a,b](x) = b(x) \leq_Q b(y) = [a,b](y)$, thus $[a,b]$ is monotone. Since $[a,b] \circ i_1 = a$ and $[a,b] \circ i_2 = b$, and these equations suffice to define $[a,b]$, uniqueness follows. □

Proposition 8, page 4: *Let $f, g \colon \langle P; \leq_P \rangle \to \langle Q; \leq_Q \rangle$ in **QOrd**. Define $E = Q/\approx$ where $\approx$ is the minimal equivalence relation containing $\{(f(x), g(x)) : x \in P\} \subseteq Q \times Q$. Also, define $e \colon Q \to E$ as $e(x) = [x]_\approx$ for all $x \in Q$. Finally, define $\leq_E$ as the reflexive and transitive closure of $\{([x]_\approx, [y]_\approx) : x \leq_Q y\}$. Then $\langle E; \leq_E \rangle$ together with e is the coequaliser of f and g in **QOrd**.*

Proof First, $\langle E; \leq_E \rangle$ is a quasi order. In fact, $[x]_\approx \leq_E [x]_\approx$ for all $x \in Q$ because $x \leq_Q x$ and $x \approx x$; also, if $[x]_\approx \leq_E [y]_\approx$ and $[y]_\approx \leq_E [z]_\approx$ then there are $z_1, \ldots, z_n, w_1, \ldots, w_m \in Q$ such that $x \leq_Q z_1 \approx z_2 \leq_Q z_3 \approx \cdots \approx z_n \leq_Q y$ and $y \leq_Q w_1 \approx w_2 \leq_Q w_3 \approx \cdots \approx w_m \leq_Q z$, so, because $y \approx y$, the sequence $z_1, \ldots, z_n, y, w_1, \ldots, w_m$ makes $[x]_\approx \leq_E [y]_\approx$. Also, if $x \leq_Q y$ then $[x]_\approx \leq_Q [y]_\approx$ by definition, so e is monotone, and, moreover $e \circ f = e \circ g$. Consider the following diagram in **QOrd**:



Let $h \colon \langle Q; \leq_Q \rangle \to \langle R; \leq_R \rangle$ in **QOrd** be such that $h \circ f = h \circ g$. Then, for all $x \in Q$ and $y, z \in [x]_\approx$, $h(y) = h(z)$. Thus, posing $k \colon \langle E; \leq_E \rangle \to \langle R; \leq_R \rangle$ as $k([x]_\approx) = h(x)$, it immediately follows that $k \circ e = h$ and this arrow is unique in **Set**. If $[x]_\approx \leq_E [y]_\approx$, then there are $z_1, \ldots, z_n$ such that $x \leq_Q z_1 \approx \cdots \approx z_n \leq_Q y$ so, being h monotone, $h(x) \leq_R h(z_1) = h(z_2) \leq_R \cdots \leq_R h(z_n) \leq_R h(y)$ and, by transitivity of $\leq_R$, $k([x]_\approx) = h(x) \leq_R h(y) = k([y]_\approx)$. □

Proposition 9, page 4: *Let $\langle P; \leq_P \rangle$ and $\langle Q; \leq_Q \rangle$ be quasi orders. Define*

$$\langle Q; \leq_Q \rangle^{\langle P; \leq_P \rangle} = \langle \{f \colon P \to Q \colon f \text{ is monotone}\}; \leq_{\mathsf{exp}} \rangle$$

where $f \leq_{\mathsf{exp}} g$ if and only if $f(x) \leq_Q g(x)$ for all $x \in P$. Also, let

$$ev \colon \langle Q; \leq_Q \rangle^{\langle P; \leq_P \rangle} \times \langle P; \leq_P \rangle \to \langle Q; \leq_Q \rangle$$

*be the function defined by $ev(f, x) = f(x)$. Then $\langle Q; \leq_Q \rangle^{\langle P; \leq_P \rangle}$ is the exponential object in **QOrd** with evaluation map ev.*

Proof It is immediate to see that $\langle Q; \leq_Q \rangle^{\langle P; \leq_P \rangle}$ is a quasi order. Also, ev is monotone: in fact, if $f \leq_{\mathsf{exp}} g$ and $x \leq_P y$, i.e., $(f, x) \leq (g, y)$ in the product, then $\mathrm{ev}(f, x) = f(x) \leq_Q f(y)$ since f is monotone, and $f(y) \leq_Q g(y)$ because $f \leq_{\mathsf{exp}} g$, so we deduce $\mathrm{ev}(f, x) \leq_Q g(y) = \mathrm{ev}(g, y)$.

Let $g \colon \langle R; \leq_R \rangle \times \langle P; \leq_P \rangle \to \langle Q; \leq_Q \rangle$ be an arrow in **QOrd**. Let $tr(g) \colon \langle R; \leq_R \rangle \to \langle Q; \leq_Q \rangle^{\langle P; \leq_P \rangle}$ defined as $\mathrm{tr}(g)(y) = \lambda y.\, g(x, y)$ for all $x \in R$. Evidently, $\mathrm{tr}(g)$ is the unique exponential transpose of g in **Set**. If $x \in R$ and $y_1 \leq_P y_2$ then $\mathrm{tr}(g)(x)(y_1) = g(x, y_1) \leq_Q g(x, y_2) = \mathrm{tr}(g)(x)(y_2)$ since g is monotone. So, for each $x \in R$, $\mathrm{tr}(g)$ is monotone. Let $x_1 \leq_R x_2$, then, for every $y \in P$, $\mathrm{tr}(g)(x_1)(y) = g(x_1, y) \leq_Q g(x_2, y) = \mathrm{tr}(g)(x_2)(y)$. Hence, $\mathrm{tr}(g)(x_1) \leq_{\mathsf{exp}} \mathrm{tr}(g)(x_2)$. □

Corollary 6, page 5: **QOrd** *is Cartesian closed and finitely cocomplete.*

Proof Immediate after Propositions 4, 5, and 9. □

Corollary 7, page 5: **Ord** *is Cartesian closed and finitely cocomplete.*

Proof Since **QOrd** is so and **Ord** is a reflective subcategory of **QOrd**, **Ord** is finitely complete and cocomplete as well. Finally, the exponential object becomes an order when $\langle Q; \leq_Q \rangle$ is so in the proof of Proposition 9. □

Proposition 10, page 6: **AOrd** *is a reflective subcategory of* **AQOrd**.

Proof It suffices to show that $\langle R; \leq_R \rangle$ as in Proposition 2 is in **AOrd** when $\langle P; \leq_P \rangle$ is in **AQOrd**. Then, the same argument as for Proposition 2 applies.

Let $\{[p_i]_\sim\}_i$ be an antichain in $\langle R; \leq_R \rangle$. Since, by definition, $[p_i]_\sim \parallel_R [p_j]_\sim$ when $i \neq j$, it follows that $p_i \parallel_P p_j$, so $\{p_i\}_i$ is an antichain in $\langle P; \leq_P \rangle$. But $\langle P; \leq_P \rangle$ has finite antichains, so $\{[p_i]_\sim\}_i$ is finite, too. □

Proposition 11, page 6: **WFOrd** *is a reflective subcategory of* **WFQOrd**.

Proof It suffices to show that $\langle R; \leq_R \rangle$ as in Proposition 2 is in **WFOrd** when $\langle P; \leq_P \rangle$ is in **WFQOrd**. Then, the same argument as for Proposition 2 applies.

Let $[p_1]_\sim \geq_R \cdots \geq_R [p_n]_\sim \geq_R \cdots$ be a proper descending chain in $\langle R; \leq_R \rangle$. By definition, $p_1 \geq_P \cdots \geq_P p_n \geq_P \cdots$, which is a descending chain in $\langle P; \leq_P \rangle$, and it is immediate to check that it is proper. But $\langle P; \leq_P \rangle$ has finite proper descending chains, being well founded, so $[p_1]_\sim \geq_R \cdots \geq_R [p_n]_\sim \geq_R \cdots$ is finite, too. □

Theorem 1, page 6: *Fixed a quasi order $\mathbb{P} = \langle P; \leq_P \rangle$, the following are equivalent:*

1. *$\mathbb{P}$ is a well quasi order;*
2. *Any infinite sequence $\{x_i\}_i$ of elements in $\mathbb{P}$ contains an increasing pair: $x_i \geq_P x_j$ for some $i < j$;*
3. *Any infinite sequence $\{x_i\}_i$ of elements in $\mathbb{P}$ contains an infinite increasing subsequence: $\left\{x_{n_j}\right\}_j$ such that $x_{n_i} \leq_P x_{n_j}$ for every $i < j$.*

Proof This is a standard result in the theory of well quasi orders [6].

First, (1) is logically equivalent to (2) since $\neg\phi \wedge \neg\psi$ is the same as $\neg(\phi \vee \psi)$, with ϕ being 'having a proper infinite descending chain', and ψ being 'having an infinite antichain'. Also, (3) trivially implies (2), and so it implies (1), too. The converse follows a Ramsey's style argument: construct the set I of indexes such that $i \in I$ has the property that, for every $j > i$, $x_j \not\geq_P x_i$. Thus, the subsequence $\{x_i\}_{i \in I}$ is such that, whenever $i < j$, either $x_i >_P x_j$ or $x_i \parallel x_j$. Consider any proper descending chain $x_{i_1} >_P \cdots >_P x_{i_n} >_P \cdots$ in $\{x_i\}_{i \in I}$: since $\mathbb{P}$ is a well quasi order, such a chain is finite. Also, one may consider just the descending chains which cannot be extended on the left, i.e., such that there is no $k < i_1$ for which $x_k >_P x_{i_1}$. Collecting in I_1 all the indexes of the first elements x_{i_1} of those 'maximal' chains, we construct a sequence $\{x_i\}_{i \in I_1}$ which forms an antichain in $\mathbb{P}$, so it must be finite. Summarising, $\{x_i\}_{i \in I}$ is finite because it can be described as a set given by the union of all the finite descending chains starting from elements in $\{x_i\}_{i \in I_1}$. Thus, any element in $\{x_i\}_i$ with an index $m > \max I$ can be used as the starting point of an infinite ascending chain. □

Proposition 12, page 7: **WFAOrd** *is a reflective subcategory of* **WFAQOrd**.

Proof Combination of Propositions 10 and 11. □

Proposition 16, page 9: *If $\langle P; \leq_P \rangle$ and $\langle Q; \leq_Q \rangle$ are both in* **WFQOrd** *or* **WFAQOrd***, then $\langle P \times Q; \leq_{P \times Q} \rangle$ is their product.*

Proof Reminding Proposition 5, it suffices to show that $\langle P \times Q; \leq_{P \times Q} \rangle$ is in the right category when $\langle P; \leq_P \rangle$ and $\langle Q; \leq_Q \rangle$ are.

First, consider when $\langle P; \leq_P \rangle$ and $\langle Q; \leq_Q \rangle$ are both in **WFQOrd**. Let $\{(p_i, q_i)\}_i$ be a proper descending chain in $\langle P \times Q; \leq_\times \rangle$. Evidently, when $i < j$, $(p_i, q_i) >_\times (p_j, q_j)$, so $p_i \geq_P p_j$ and $q_i \geq_Q q_j$. Thus $\{p_i\}_i$ and $\{q_i\}_i$ are descending chains in $\langle P; \leq_P \rangle$ and $\langle Q; \leq_Q \rangle$, respectively.

If $\{r_i\}_i$ is a descending chain in $\langle R;\leq_R\rangle \in \mathrm{Obj}\,\mathbf{WFQOrd}$, then $\{r'_i\}_i$, where r'_k is the least element in the chain $\{r_i\}_i$ such that $r'_k \leq_R r'_h$ but $r'_h \not\leq_R r'_k$ for every $h < k$, is evidently a proper descending chain.

Hence, $\{p'_i\}_i$ and $\{q'_i\}_i$ are proper descending chains in $\langle P;\leq_P\rangle$ and $\langle Q;\leq_Q\rangle$, respectively. But $\langle P;\leq_P\rangle$ and $\langle Q;\leq_Q\rangle$ are in **WFQOrd** so $\{p'_i\}_i$ and $\{q'_i\}_i$ are both finite. Then $\{(p'_i,q'_i)\}_i$ is finite, too.

Since the construction of the chain $\{r'_i\}_i$ out of $\{r_i\}_i$ proceeds by 'compressing' equivalent consecutive elements to the first one appearing in the chain $\{r_i\}_i$, we can consider the subsequences

$$\left(p'_k, q_{n_0}\right), \ldots, \left(p_{n_i}, q_{n_i}\right), \ldots$$

such that $p'_k \sim p_{n_i}$, i.e., such that the first components are equivalent. Since $\{(p_i,q_i)\}_i$ is a proper descending chain, posing $p_{n_0} = p'_k$, $(p_{n_i},q_{n_i}) \geq_\times \left(p_{n_j},q_{n_j}\right)$ whenever $i < j$ in the subsequence, thus $q_{n_i} >_Q q_{n_j}$ whenever $i < j$, that is, $\left\{q_{n_i}\right\}_i$ is a proper descending chain in $\langle Q;\leq_Q\rangle$, so it must be finite. Hence, the selected subsequence must be finite, too. Of course, symmetrically, we obtain finite subsequences when considering chains of elements having the first component equivalent in $\langle Q;\leq_Q\rangle$.

Therefore, $\{(p_i,q_i)\}_i$ can be divided into a finite sequence $\{(p''_i,q''_i)\}_i$, with p''_i equivalent to p'_i and q''_i equivalent to q'_i, of finite subsequences as above, which forces $\{(p_i,q_i)\}_i$ to be finite. So, $\langle P\times Q;\leq_{P\times Q}\rangle$ is in **WFQOrd**.

When $\langle P;\leq_P\rangle$ and $\langle Q;\leq_Q\rangle$ are both in **WFAQOrd**, the result is already well-known in literature, and it can be shown to be valid by applying Dickson's lemma [3], or the infinite Ramsey's theorem [5]. □

Proposition 17, page 9: *Let $f,g\colon \langle P;\leq_P\rangle \to \langle Q;\leq_Q\rangle$ be in* **AQOrd**, **WFQOrd**, *or* **WFAQOrd**.

Define $E = \{x \in P\colon f(x) = g(x)\}$, and $\leq_E$ as the restriction of $\leq_P$ to $E\times E$. Then, $\langle E;\leq_E\rangle$ and the inclusion $i\colon E \to P$ is the equaliser of f and g in **AQOrd**, **WFQOrd**, *and* **WFAQOrd**, *respectively.*

Proof After Proposition 6, it suffices to show that $\langle E;\leq_E\rangle$ is in the same category as f and g. If $\langle P;\leq_P\rangle$ has finite antichains, consider any antichain $\{e_i\}_i$ in $\langle E;\leq_E\rangle$: since the inclusion map i is monotone, $\{e_i\}_i$ is an antichain in $\langle P;\leq_P\rangle$ thus it is finite.

Also, if $\langle P;\leq_P\rangle$ has finite proper descending chains, consider any proper descending chain $\{e_i\}_i$ in $\langle E;\leq_E\rangle$: since the inclusion map i is monotone, $\{e_i\}_i$ is a proper descending chain in $\langle P;\leq_P\rangle$, too, thus it is finite. □

Proposition 18, page 9: *If $\langle P;\leq_P\rangle$ and $\langle Q;\leq_Q\rangle$ are both in* **AQOrd**, **WFQOrd**, *or* **WFAQOrd**, *then their coproduct is $\langle P\sqcup Q;\leq_{P\sqcup Q}\rangle$.*

Proof Reminding Proposition 7, it suffices to show that $\langle P\sqcup Q;\leq_{P\sqcup Q}\rangle$ is in the right category when $\langle P;\leq_P\rangle$ and $\langle Q;\leq_Q\rangle$ are.

If $X = \{x_i\}_i$ is an antichain in $\langle P\sqcup Q;\leq_{P\sqcup Q}\rangle$, then $X_P = \{x_i \in P\}_i$ and $X_Q = \{x_i \in Q\}_i$ are antichains in $\langle P;\leq_P\rangle$ and $\langle Q;\leq_Q\rangle$, respectively. So, when these quasi orders have finite antichains, $X = X_P \sqcup X_Q$ is finite too.

If $X = \{x_i\}_i$ is a proper descending chain in $\langle P\sqcup Q;\leq_{P\sqcup Q}\rangle$, then, by definition of $\leq_{P\sqcup Q}$, X is a proper descending chain either in $\langle P;\leq_P\rangle$ or in $\langle Q;\leq_Q\rangle$. So, when these quasi orders are well founded, X must be finite, too. □

Proposition 19, page 9: **AQOrd** *has coequalisers.*

Proof By Proposition 8, and following the same notation, it suffices to show that $\langle E;\leq_E\rangle \in \mathrm{Obj}\,\mathbf{AQOrd}$, when $\langle P;\leq_P\rangle$ and $\langle Q;\leq_Q\rangle$ are. Let $\{[e_i]_\approx\}_i$ be an antichain in $\langle E;\leq_E\rangle$. When $e_i <_Q e_j$, then $[e_i]_\approx \leq_E [e_j]_\approx$, which is impossible; also, if $e_i \geq_Q e_j$, then $[e_i]_\approx \geq_E [e_j]_\approx$, which is impossible, too. Hence, $e_i \parallel e_j$ whenever $i \neq j$, that is $\{e_i\}_i$ is an antichain in $\langle Q;\leq_Q\rangle$, so it is finite, forcing $\{[e_i]_\approx\}_i$ to be finite, too. □